Modelling Fluid Mechanics

Mathematics Session 7

T J James

Reach 2023

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Schedule

1. ODE Solving Packages
   - First Order ODEs
   - Second Order ODEs
2. Simple Harmonic Motion
   - Modelling Waves
3. Further Modelling
   - Building Models
     - Stokes' Law
     - Buoyancy
     - Bernoulli's Principle

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ODE Solving Packages

First Order ODEs

ODE45 Recap

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dydt = @(t,y) 2*t;

[t, y] = ode45(dydt, time, y0);
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Simple ODE45 Simulation

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time = [0, 5];
y0 = 0;

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plot(t, y)
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2. Build a simulation using ODE45 which models the solution to Equation (1) for $t \in [-1, 1]$.
   Hint: You can't just set `y0 = 2`*, you need to work in two halves.*

ODE Solving Packages

Second Order ODEs

Coupled Equations

ODE45 can only find solutions for first-order ODEs, since it can only solve functions of the form

$$\frac{dy}{dt} = y'(t) = f(t, y).$$

But we may want to solve a second-order ODE, we would have an equation where

$$\frac{d^2y}{dt^2} = y''(t) = g(t, y, y').$$

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So using these, we end up with two equations, where,

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$$\begin{aligned} y_1' &= y_2, \\ y_2' &= g(t, y_1, y_2). \end{aligned}$$

Solving Second-Order ODEs

Consider the following initial value problem,

$$y'' - y' + 3y = t,$$

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Rewrite the problem into $y''(t) = g(t, y, y')$,

$$y'' = t + y' - 3y$$

Now use $y_1(t) = y(t)$, $y_2(t) = y'(t)$, so

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= t + y_2 - 3y_1, \end{aligned}$$

where $y_1(t = 0) = 1$ and $y_2(t = 0) = 2$.

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$$\texttt{[1, 2, 3]} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad \texttt{[1; 2; 3]} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

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plot(ts, ys(:, 1))
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- 1 - The number means that we only want the first column of the array.

Simulating Second-Order ODEs

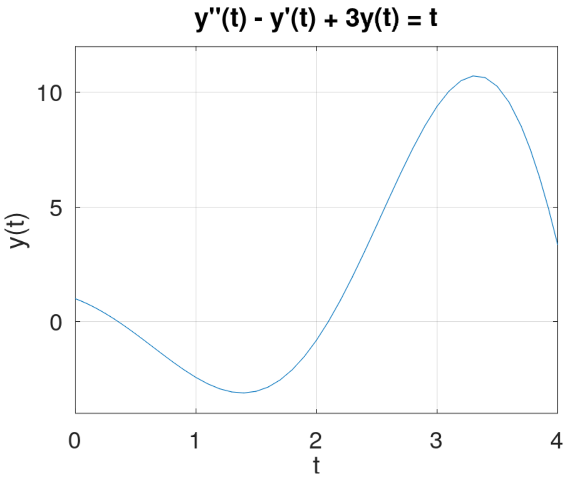
```
time = [0, 4];
y0 = [1; -2];

dydt = @(t,y) [y(2); t + y(2) - 3 * y(1)];

[ts, ys] = ode45(dydt, time, y0);

plot(ts, ys(:,1))
hold on

grid on;
title("y''(t) - y'(t) + 3y(t) = t");
xlabel("t");
ylabel("y(t)");
axis([0 4 -4 12]);
```



Solving a Second-Order ODE

Task

Build a simulation using ODE45 which models the solution to Equation (2) for $t \in [0, \pi]$.

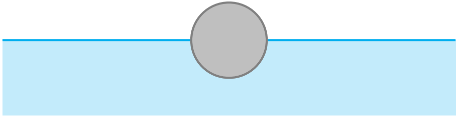
$$y'' + 2y = \sin(x), \quad y(t=0) = 0, \; y'(t=0) = -1 \tag{2}$$

Simple Harmonic Motion

Modelling Waves

Floating Ball

- Consider a ball floating in water.



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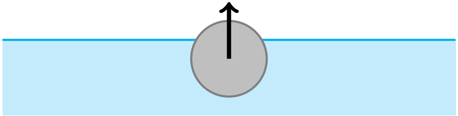

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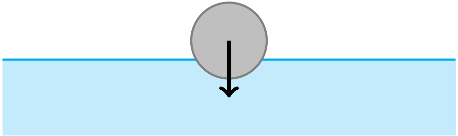

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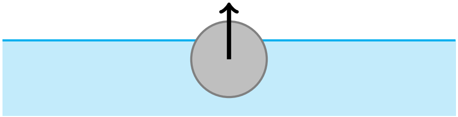

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- If we assume no friction/resistance, the ball will continue to bob up and down indefinitely.

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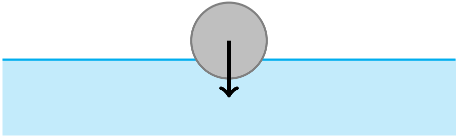

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  - This force will push the ball up.
- Once it's above the water-line, it will feel a downwards force from gravity.
- If we assume no friction/resistance, the ball will continue to bob up and down indefinitely.
- This is **simple harmonic motion**.
  - A motion in which the restoring force is directly proportional to the displacement of the body from its equilibrium position.

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- And so, our final equation is,

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- And lastly, we should specify some initial conditions,

$$y(t=0) = L, \quad \frac{dy}{dt}(t=0) = 0.$$

Simulating Simple Harmonic Motion

Theorem

Simple harmonic motion is quantified by,

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Task

1. Solve the SHM equation.
2. Build a simulation using ODE45 which models SHM between $t \in [0, 10]$.
   For the constants, use $L = 1$ and $m = 1$, and water has a spring constant of $k \approx 20$.

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- With some rearrangement, we get,

$$\frac{d^2y}{dt^2} = -\frac{c}{m}\frac{dy}{dt} - \frac{k}{m}y.$$

- And we can keep the same initial conditions,

$$y(t=0) = L, \quad \frac{dy}{dt}(t=0) = 0.$$

Simulating Damped Simple Harmonic Motion

Theorem

Damped simple harmonic motion is quantified by,

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Task

1. Modify your simulation from earlier to model damped SHM between $t \in [0, 10]$.
   Using the same constants as before ($L = 1$, $m = 1$, $k = 20$), using a damping coefficient of $c = 0.5$.

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Task

1. Modify your simulation from earlier to model damped SHM between $t \in [0, 10]$.
   Using the same constants as before ($L = 1$, $m = 1$, $k = 20$), using a damping coefficient of $c = 0.5$.
2. Now run your simulation for each of the following, and comment on what you notice.
   - $c = 1$
   - $c = 2$
   - $c = 5$
   - $c = 10$
   - $c = 20$
   - $c = 50$

Further Modelling

Building Models

Instructions

Over to you!

On the following slides, details on how to begin modelling each of the earlier experiments are given.

- Archimedes' Principle
- Stokes' Theorem
- Buoyancy
- Bernoulli's Principle

Using the information on these slides and your knowledge of Octave (along with whatever you can find out online), your job is to produce a simulation of the experiments from previous sessions.

Focus mainly on your experiment, but you can model some of the other experiments as well if you wish.

If you have a working simulation which matches your results, then there are some extension ideas for each one.

Further Modelling
Building Models - Stokes' Law

Stokes' Law

The viscoscity of a fluid is given by,

$$F = 6\pi\mu r v.$$

Force on object at terminal velocity is,

$$F = mg - \rho g \frac{4}{3}\pi r^3.$$

Constants

F	Force on an object
μ	Viscosity
r	Radius of spherical object
v	Terminal velocity
m	Mass of object
g	Gravitational constant $= 9.81$ ms^{-1}
ρ	Density of fluid

Stokes' Law

Simulation Suggestion

Plot the graph comparing the terminal velocity of a sphere relative to the viscosity of the fluid it falls through.

Extension Ideas

- Use the Buoyancy experiment to consider how Stokes' Law could be used to find the salinity of water by finding its density first.
- Consider differently shaped objects and how Stokes' Law might apply to them. For example, how would a cube or a cone behave?

Further Modelling

Building Models - Buoyancy

Buoyancy

The density of water is directly proportional to its salinity.
The salinity of water is given by,

$$S = \frac{m_{\text{salt}}}{m_{\text{salt}} + m_{\text{water}}}$$

Salinity/ %	Density/ $\text{kg} \cdot \text{m}^{-3}$
0	998.34
5	1036.39
10	1075.58
15	1116.36
20	1158.89

Constants

S	Salinity
$m_{\text{salt/water}}$	Mass of salt/water

Buoyancy

Simulation Suggestion

Plot the graph comparing the density of water relative to its salinity, finding the linear relationship between them. Then use compare this to the average density of your egg.

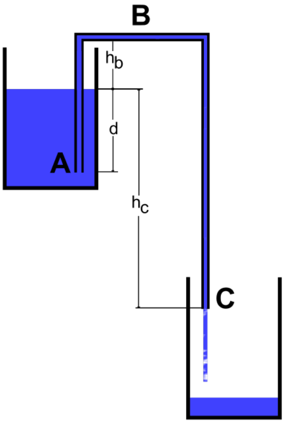
Extension Ideas

- Use Stokes' Law to calculate the speed at which the egg falls through water.
- Consider items with different densities and compare what the salinity of the water would need to be in order for them to float in the same context.

Further Modelling

Building Models - Bernoulli's Principle

Bernoulli's Principle



Bernoulli's equation for a siphon is,

$$\frac{v^2}{2} + gy + \frac{P}{\rho} = \text{Constant.}$$

One can find four equations for the system.

Location	Equation
$y = 0$	$\frac{0^2}{2} + g(0) + \frac{P_{atm}}{\rho} = \text{Const}$
A	$\frac{v_A^2}{2} - gd + \frac{P_A}{\rho} = \text{Const}$
B	$\frac{v_B^2}{2} + gh_B + \frac{P_B}{\rho} = \text{Const}$
C	$\frac{v_C^2}{2} - gh_C + \frac{P_{atm}}{\rho} = \text{Const}$

By equating these equations, one can derive formulae for v_{max}, h_{max}, etc.

Water is incompressible, so $v_A = v_B = v_C$.

Constants

v	Fluid velocity along streamline
g	Gravitational constant $= 9.81$ ms^{-1}
y	Elevation
P	Pressure along streamline
ρ	Fluid density

Bernoulli's Principle

Simulation Suggestion

Plot the graph comparing pressure throughout the siphon relative to its y-value.

Extension Ideas

- Plot the graph based on the distance travelled along the siphon, rather than simply the y-value.
- Derive equations for v_{max}, h_{max}, etc, and compare how varying certain aspects of the siphon might change these things.